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A POSTERIORI ERROR ESTIMATION "NEW" APPROACH

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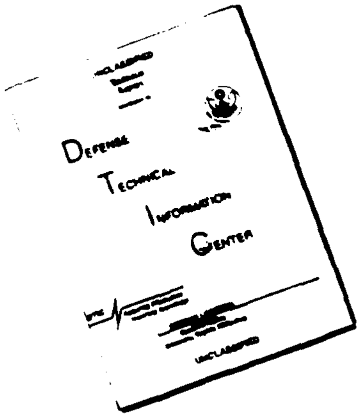

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A POSTERIORI ERROR ESTIMATION

"NEW" APPROACH

PATRICK J. RABIER

1. Preliminaries.

Let V be a real Hilbert space and $a : V \times V \longrightarrow \mathbf{R}$ a continuous bilinear form. It is a more or less standard result that if a satisfies the conditions

$$\inf_{v \neq 0} \sup_{w \neq 0} \frac{a(v,w)}{\|v\|\,\|w\|} \geq c > 0, \tag{1.1a}$$

where c is a constant, and

$$\{a(v,w) = 0, \ \ \forall v \in V\} \Rightarrow w = 0, \tag{1.1b}$$

then there is $L \in GL(V)$, unique, such that

$$a(v,w) = (Lv, w), \tag{1.2}$$

and hence given $\ell \in V'$, there is a unique solution $u \in V$ of the problem

$$a(u,v) = \ell(v), \quad \forall v \in V. \tag{1.3}$$

Indeed, existence of $L \in \mathcal{L}(V)$ in (1.2) follows from the Riesz representation theorem, and (1.1a) easily implies that L is one-to-one with closed range. Next, from closedness of rge L and the relation rge $L = (\ker\ L^*)^{\perp}$, and since condition (1.1b) implies $\ker L^* = \{0\}$, we infer that L is onto V, i.e. $L \in GL(V)$ by Banach's theorem. Next, if g represents ℓ via the Riesz representation theorem, that is, $\ell(v) = (g,v)$, $\forall v \in V$, $u = L^{-1}g$ is the unique solution of (1.3).

Conditions (1.1a/b) include, but are not limited to, the case when a satisfies the hypothesis of the Lax-Milgram lemma

From now on, we assume that $V = H_0^1(\Omega)$ where $\Omega \subset \mathbf{R}^n$ is a bounded open subset with Lipschitz continuous boundary Γ.

Let $(\Omega_K)_{1 \le K \le N}$ be a collection of open subsets of Ω with Lipschitz continuous boundaries Γ_K, and such that $\Omega_K \cap \Omega_L = \emptyset$ if $K \ne L$ and $\bigcup_{K=1}^{N} \bar{\Omega}_K = \bar{\Omega}$. For convenience, we shall refer to (Ω_K) as a "partition" $\mathcal{P}$ of Ω.

We shall denote by $\mathcal{I}$ the set of pairs (K, L) with $K \ne L$ such that meas $(\Gamma_K \cap \Gamma_L) > 0$. Here, "meas" refers to the Γ_K-, or equivalently Γ_L- Lebesque measure. Also, we set

$$\mathcal{I}_K = \{1 \le L \le N : (K, L) \in \mathcal{L}\}, \quad 1 \le K \le N,$$

and

$$\Gamma_{KL} = \Gamma_K \cap \Gamma_L, \quad \forall (K, L) \in \mathcal{I}.$$

Let $\Gamma_{\mathcal{I}}$ be the *disjoint* union

$$\Gamma_{\mathcal{I}} = \coprod_{(K,L) \in \mathcal{I}} \Gamma_{KL}$$

(hence, in $\Gamma_{\mathcal{I}}, \Gamma_{KL}$ and Γ_{LK} are different although they coincide as subsets of $\mathbf{R}^n$).

We introduce the space

$$L^2(\Gamma_{\mathcal{I}}) = \prod_{(K,L) \in \mathcal{I}} L^2(\Gamma_{KL}).$$

Elements of $L^2(\Gamma_{\mathcal{I}})$ thus are collections $\lambda = (\lambda_{KL})$ with $\lambda_{KL} \in L^2(\Gamma_{KL})$. Note that the functions λ_{KL} and λ_{LK} are generally different, despite the fact that they are both defined in the same subset $\Gamma_K \cap \Gamma_L$ of $\mathbf{R}^n$.

It is clear that there is a canonical identification

$$L^2(\Gamma_{\mathcal{I}} \simeq \prod_{K=1}^{N} L^2(\Gamma_K \setminus \Gamma)$$

obtained by associating with $\lambda = (\lambda_{KL}) \in L^2(\Gamma_{\mathcal{I}})$ the collection (λ_K) defined by

$$\lambda_{K|\Gamma_{KL}} = \lambda_{KL}, \quad \forall L \in \mathcal{I}_K,$$

and vice-versa.

In future considerations, it will be convenient to use both notations (λ_{KL}) and (λ_K) for elements of $L^2(\Gamma_{\mathcal{I}})$. No confusion should arise from this since the number of indices used immediately identifies which definition of $L^2(\Gamma_{\mathcal{I}})$ is being used.

The norm of $L^2(\Gamma_{\mathcal{I}})$ is the natural one, namely

$$|\lambda|_{0,\Gamma_{\mathcal{I}}} = \left(\sum_{(K,L)\in\mathcal{I}} |\lambda_{KL}|^2_{0,\Gamma_{KL}} \right)^{1/2} = \left(\sum_{K=1}^{N} |\lambda_K|^2_{0,\Gamma\setminus\Gamma_K} \right)^{1/2}.$$

We shall also consider the spaces

$$V_K = \{v_K \in H^1(\Omega_K) : v_K = 0 \text{ in } \Gamma\cap\Gamma_K\},$$

with norm induced by the norm of $H^1(\Omega_K)$, and their product

$$H_0^1(\mathcal{P}) = \prod_{K=1}^{N} V_K,$$

equipped with the product norm, denoted by $\|\cdot\|_{1,\mathcal{P}}$.

Elements V of $H_0^1(\mathcal{P})$ can be viewed as N-tuples (V_K) with $v_K \in V_K$ or as functions in Ω (defined almost everywhere) such that $v_K = v_{|\Omega_K} \in V_K$. Both points of view will be used later, and once again no confusion should arise from this.

A restriction map (trace) is defined:

$$v \in H_0^1(\mathcal{P}) \longmapsto (v_{K_{|\Gamma_K\setminus\Gamma}}) \in L^2(\Gamma_{\mathcal{I}}),$$

which is evidently continuous for the norm of the spaces involved.

There is a canonical embedding

$$H_0^1(\Omega) \hookrightarrow H_0^1(\mathcal{P})$$

defined by $v \longmapsto (v_K)$ where $v_K = v_{|\Omega_K}$. On the other hand, the space $L^2(\Gamma_{\mathcal{I}})$ can be split into the direct sum

$$L^2(\Gamma_{\mathcal{I}}) = \Lambda_+ \oplus \Lambda_-,$$

where

$$\Lambda_+ = \{(\Gamma_{KL}) : \Gamma_{KL} = \Gamma_{LK}, \quad \forall (K,L) \in \mathcal{I}\}$$

and

$$\Lambda_- = \{(\Gamma_{KL}) : \Gamma_{KL} = -\Gamma_{LK}, \quad \forall (K,L) \in \mathcal{I}\}.$$

The following result characterizes the element of $H_0^1(\Omega)$ among the elements of $H_0^1(\mathcal{P})$:

Proposition 1.1: Let $v = (v_K) \in H_0^1(\mathcal{P})$. Then, $v \in H_0^1(\Omega)$ if and only if $(v_{K|_{\Gamma_K \setminus \Gamma}}) \in \Lambda_+$.

Proof: If $v \in \mathcal{D}(\Omega)$ and $v_K = v_{|_{\Omega_K}}$, it is obvious that $v_{K|_{\Gamma_{KL}}} = v_{L|_{\Gamma_{KL}}}$, i.e. $(v_{K|_{\Gamma_K \setminus \Gamma}}) \in \Lambda_+$. By denseness of $\mathcal{D}(\Omega)$ into $H_0^1(\Omega)$, continuity of the trace $H_0^1(\mathcal{P}) \to L^2(\Gamma_{\mathcal{I}})$ and closedness of Λ_+ in $L^2(\Gamma_{\mathcal{I}})$, this relation extends to the whole space $H_0^1(\Omega)$.

Conversely, we have $H_0^1(\mathcal{P}) \hookrightarrow L^2(\Omega)$ in the obvious way, and for $\varphi \in \mathcal{D}(\Omega)$ and $v \in H_0^1(\mathcal{P})$, we have

$$\langle \frac{\partial v}{\partial x_i}, \varphi \rangle = -\langle v, \frac{\partial \varphi}{\partial x_i} \rangle = -\sum_{K=1}^{N} \int_{\Omega_K} v_K \frac{\partial \varphi}{\partial x_i} = \sum_{K=1}^{N} \int \frac{\partial v_K}{\partial x_i} \varphi - \sum_{K=1}^{N} \int_{\Gamma_K} v_K \varphi \nu_{K_i},$$

where $\nu_K = (\nu_{K1}, \cdots, \nu_{Kn})$ is the outward unit normal vector along Γ_K. Since $\varphi = 0$ on Γ, we find

$$\sum_{K=1}^{N} \int_{\Gamma_K} v_K \varphi \nu_{K_i} = \sum_{(K,L) \in \mathcal{I}} \int_{\Gamma_{KL}} v_K \varphi \nu_{K_i},$$

and the terms $\int_{\Gamma_{KL}} v_L \varphi \nu_{L_i}$ cancel out from the hypothesis $v_K = v_L$ on Γ_{KL} since $\nu_{K_i} = -\nu_{L_i}$. It follows that

$$\langle \frac{\partial v}{\partial x_i}, \varphi \rangle = \sum_{K=1}^{N} \int_{\Omega_K} \frac{\partial v_K}{\partial x_i} \varphi.$$

i.e. $\partial v / \partial x_i$ is represented by the $L^2(\Omega)$ function defined by $\partial v_K / \partial x_i$ in Ω_K, $1 \le K \le N$. Thus, $v \in H^1(\Omega)$, and since $v = 0$ in $\bigcup_{K=1}^{N} (\Gamma_K \cap \Gamma) = \Gamma$, we have $v \in H_0^1(\Omega)$. □

2. The space Λ_-.

Every element $\lambda \in \Lambda_-$ can be viewed as an element of $(H_0^1(\mathcal{P}))' = \prod_{K=1}^{N} V_K'$ via

$$\langle \lambda, w \rangle = \sum_{K=1}^{N} \int_{\Gamma_K \setminus \Gamma} \lambda_K w_K$$

note that

(2.1) $$\langle \lambda, w \rangle = 0, \quad \forall w \in H_0^1(\Omega), \quad \forall \lambda \in \Lambda_-,$$

since

$$\sum_{K=1}^{N} \int_{\Gamma_K \setminus \Gamma} \lambda_K w_K = \sum_{(K,L) \in \mathcal{I}} \int_{\Gamma_{KL}} \lambda_K w_K,$$

and the terms $\int_{\Gamma_{KL}} \lambda_K w_K$ and $\int_{\Gamma_{KL}} \lambda_L w_L$ cancel out because $w_K = w_L$ on Γ_{KL} (see Proposition 1.1) and $\lambda_K = -\lambda_L$ on Γ_{KL} if $\lambda \in \Lambda_-$.

Definition 2.1: The space $\bar{\Lambda}_-$ is the closure of Λ_- in $(H_0^1(\mathcal{P}))' = \prod_{K=1}^{N} V_K'$.

By definition, every $\lambda \in \bar{\Lambda}_-$ is the limit in $(H_0^1(\mathcal{P}))'$ of a sequence $\lambda^j \in \Lambda_-$. Since $\langle \lambda^j, w \rangle = 0$, $\forall w \in H_0^1(\Omega)$ from (2.1), we obtain by continuity that

(2.2) $$\langle \lambda, w \rangle = 0, \quad \forall \lambda \in \bar{\Lambda}_-, \quad \forall w \in H_0^1(\Omega).$$

Let $A = (a_{ij}) \in (L^\infty(\Omega))^{n \times n}$ and let $H_{0,A}^1(\Omega)$ be the subspace

$$H_{0,A}^1(\Omega) = \{v \in H_0^1(\Omega) : \nabla \cdot A \nabla v \in L^2(\Omega)\},$$

equipped with the norm

$$\|v\|_{1,A,\Omega} = (\|v\|_{1,\Omega}^2 + |\nabla \cdot A \nabla v|_{0,\Omega}^2)^{1/2}.$$

Proposition 2.1: The mapping

$$v \in \mathcal{D}(\Omega) \longmapsto \left(\frac{\partial v_K}{\partial \nu_A} \right) = (A \nabla v_K \cdot \nu_K) \in \Lambda_-$$

(where $v_K = v_{|\Omega_K}$, $1 \le K \le N$) can be extended as a linear continuous mapping from $H_{0,A}^1(\Omega)$ into $\bar{\Lambda}_-$.

Proof: It is plain that $(\partial v_K/\partial \nu_A) \in \Lambda_-$ as soon as $v \in \mathcal{D}(\Omega)$. In addition, by Green's formula,

$$\int_{\Gamma_K} \frac{\partial v_K}{\partial \nu_A} w_K = \int_{\Omega_K} \nabla A v_K \cdot \nabla w_K + \int_{\Omega_K} \nabla \cdot (A \nabla v_K) w_K, \quad \forall w_K \in V_K.$$

Thus, adding up these relations, we find

$$|\langle \left(\frac{\partial v_K}{\partial \nu_A}\right), w \rangle | \leq C \|v\|_{1,A,\Omega} \, \|w\|_{1,\mathcal{P}}, \quad \forall w \in H_0^1(\mathcal{P}),$$

where $C > 0$ is a constant depending only upon A. Hence,

$$\| \left(\frac{\partial v_K}{\partial v_A}\right) \|_{*,\mathcal{P}} \leq C \|v\|_{1,A,\Omega}, \quad \forall v \in \mathcal{D}(\Omega),$$

where $\| \cdot \|_{*,\mathcal{P}}$ denote the norm in $(H_0^1(\mathcal{P}))'$. That $(\partial v_K/\partial \nu_A)$ can be extended as an element of $\bar{\Lambda}_-$ for $v \in H_{0,A}^1(\Omega)$ follows from this inequality and denseness of $\mathcal{D}(\Omega)$ into $H_{0,A}^1(\Omega)$ (for the $\| \cdot \|_{1,A,\Omega}$ norm) by standard arguments. □

Remark 2.1: Because $\bar{\Lambda}_- \subset (H_0^1(\mathcal{P}))' = \prod_{K=1}^{N} V_K'$, elements of $\bar{\Lambda}_-$ are N-tuples $\lambda = (\lambda_K)$ with $\lambda_K \in V_K'$ but unlike elements of Λ_-, they cannot be identified with collections (λ_{KL}) for $(K, L) \in \mathcal{I}$ (since elements of V_K' are not functions, they cannot be restricted to subsets). □

3. Application to error estimation in linear problems.

In addition to the coefficients $A = (a_{ij}) \in (L^\infty(\Omega))^{n \times n}$ of the previous section, we also consider $b = (b_i) \in (L^\infty(\Omega))^n$ and $c \in L^\infty(\Omega)$, and set

$$a(v, w) = \int_\Omega A \nabla v \cdot \nabla w + \int_\Omega (b \cdot \nabla v) w + \int_\Omega c v w, \quad \forall v, w \in H_0^1(\Omega).$$

We shall henceforth assume that $a(\cdot, \cdot)$ above satisfies the conditions (1.1a/b). Among other things, this ensures that given $f \in L^2(\Omega)$, there is a unique solution $u \in H_0^1(\Omega)$ of the problem

$$a(u, v) = \int_\Omega f v, \quad \forall v \in H_0^1(\Omega).$$

Observe that u solves the PDE

$$-\nabla \cdot (A\nabla u) + b \cdot \nabla u + cu = f \quad \text{in } \Omega, \tag{3.1}$$

whence $\nabla \cdot (A\nabla u) = b \cdot \nabla u + cu - f \in L^2(\Omega)$, i.e. $u \in H^1_{0,A}(\Omega)$.

For $1 \le K \le N$, let $a_K : V_K \times V_K \to \mathbf{R}$ be the (continuous) bilinear form

$$a_K(v_K, w_K) = \int_{\Omega_K} A\nabla v_K \cdot \nabla w_K + \int_{\Omega_K} (b \cdot \nabla v_K) w_K + \int_{\Omega_K} c v_K w_K, \quad \forall v_K, w_K \in V_K.$$

We assume that each bilinear form a_K satisfies the conditions (1.1a/b), so that given $\hat{u} \in H^1_0(\Omega)$ the problem

$$a_K(\phi_K, v_K) = \int_{\Omega_K} f v_K = a_K(\hat{u}, v) + \langle \lambda_K, v_K \rangle, \quad \forall v_K \in V_K \tag{3.2}$$

has a unique solution $\phi_K(\lambda_K) \in V_K$ for every $\lambda_K \in V_K'$. In particular, given $\lambda = (\lambda_K) \in \tilde{\Lambda}_-$, there is a unique $\phi(\lambda) \in H^1_0(\mathcal{P})$ such that $\phi_K = \phi_K(\lambda_K)$ solves (3.2) for $1 \le K \le N$. The following theorem shows that the error $u - \hat{u}$ is characterized by a property of $\phi(\lambda)$ holding for one and only one $\lambda \in \tilde{\Lambda}_-$.

Theorem 3.1: The following conditions are equivalent:
(i) $\phi(\lambda) \in H^1_0(\Omega)$,
(ii) $(\phi_K(\lambda_K)_{|\Gamma_K \Gamma}) \in \Lambda_+$,
(iii) $\phi(\lambda) = u - \hat{u}$,
(iv) $\lambda = (\partial u_K / \partial \nu_A)$.

Proof: For (i) $\Leftrightarrow$ (ii), see Proposition 1.1. Before proving that (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv), let us note that from (3.1) and Green's formula, we have

$$\int_{\Omega_K} f v_K = a_K(u, v_K) - \langle \frac{\partial u_K}{\partial \nu_A}, v_K \rangle,$$

where $\partial u_K / \partial \nu_A \in V_K'$ vanishes in $H^1_0(\Omega_K)$ (the validity of Green's formula above is due to $u \in H^1_{0,A}(\Omega)$). By substitution into (3.2), we get

$$a_K(\phi_K(\lambda_K), v_K) = a_K(u - \hat{u}, v_K) + \langle \lambda_K - \frac{\partial u_K}{\partial \nu_A}, v_K \rangle, \quad \forall v_K \in V_K, \tag{3.3}$$

and hence

$$\sum_{K=1}^{N} a_K(\phi_K(\lambda_K), v_K) = \sum_{K=1}^{N} a_K(u - \hat{u}, v_K) + \sum_{K=1}^{N} (\lambda_K - \frac{\partial u_K}{\partial \nu_A}, v_K), \ \forall v \in H_0^1(\mathcal{P}).$$

(i) $\Rightarrow$ (iii): Choose $v \in H_0^1(\Omega)$ in the above relation. From (2.2) and $\lambda, (\partial u_K/\partial \nu_A) \in \hat{\Lambda}_-$, it follows that

$$\sum_{K=1}^{N} a_K(\phi_K(\lambda_K), v_K) = \sum_{K=1}^{N} a_K(u - \hat{u}, v_K), \quad \forall v \in H_0^1(\Omega).$$

But $u - \hat{u} \in H_0^1(\Omega)$ and $\phi(\lambda) \in H_0^1(\Omega)$ from (i), so that the above equality reads

$$a(\phi(\lambda), v) = a(u - \hat{u}, v), \quad \forall v \in H_0^1(\Omega),$$

i.e. $\phi(\lambda) = u - \hat{u}$ since $a(\cdot,\cdot)$ satisfies (1.1a/b).

(iii) $\Rightarrow$ (iv): If $\phi(\lambda) = u - \hat{u}$, then $\phi_K(\lambda_K) = (u - \hat{u})_{|\Omega_K}$ and (3.3) yields

$$(\lambda_K - \frac{\partial u_K}{\partial \nu_A}, v_K) = 0, \quad \forall v_K \in V_K, \ 1 \le K \le N.$$

As a result, $\lambda_K = \partial u_K/\partial \nu_A$ (equality in V_K'), $1 \le K \le N$, i.e. $\lambda = (\partial u_K/\partial \nu_A)$.

(iv) $\Rightarrow$ (i): If $\lambda = (\partial u_K/\partial \nu_A)$, then $\lambda_K = \partial u_K/\partial \nu_A, 1 \le K \le N$, and (3.3) shows that $\phi_K(\lambda_K)$ is characterizeed by

$$a_K(\phi_K(\lambda_K), v_K) = a_K(u - \hat{u}, v_K), \quad \forall v_K \in V_K.$$

But certainly $\phi_K(\lambda_K) = (u - \hat{u})_{|\Omega_K}$ satisfies the above relation, and hence is the unique solution of (3.3). This shows that $\phi(\lambda) = u - \hat{u} \in H_0^1(\Omega)$. □

Corollary 3.1: The functional

$$\lambda \in \Lambda_- \longmapsto J(\lambda) = \frac{1}{2} \sum_{\substack{(K,L)\in \mathcal{I} \\ K>L}} |\phi_K(\lambda_K) - \phi_L(\lambda_L)|_{0,\Gamma_{KL}}^2$$

has the unique minimizer $\lambda = (\partial u_K/\partial \nu_A)$, for which $\phi(\lambda) = u - \hat{u}$.

Proof: From (iv) $\Rightarrow$ (ii) in Theorem 3.1, we have $J((\partial u_K/\partial \nu_A)) = 0$, and hence $\lambda = (\partial u_K/\partial \nu_A)$ is a minimizer of J since $J \geq 0$. This also shows that $J(\lambda) = 0$ for any minimizer of J but then $\lambda = (\partial u_K/\partial \nu_A)$ from (ii) $\Rightarrow$ (iv) in Theorem 3.1. Thus, $\lambda = (\partial u_K/\partial \nu_A)$ is the unique minimizer of J, and $\phi(\lambda) = u - \hat{u}$ by (iv) $\Rightarrow$ (iii) in Theorem 3.1. $\square$

Corollary 3.1 suggests a strategy to calculate the error $u - \hat{u}$, by minimizing the functional J in (3.4). It is important to notice that the functional J is quadratic. More specifically, denote by $\beta_K \in V_K$ the (unique) solution of

$$a_K(\beta_K, v_K) = \int_{\Omega_K} f v_K - a_K(\hat{u}, v_K), \quad \forall v_K \in V_K,$$

and let $U_K \in \mathcal{L}(V_K', V_K)$ be defined by

$$a_K(U_K v_K^*, v_K) = (v_K^*, v_K), \quad \forall v_K^* \in V_K', \quad \forall v_K \in V_K.$$

Then, $\phi_K(\lambda_K) = U_K \lambda_K + \beta_K$, $1 \leq K \leq N$, for $\lambda \in \Lambda_-$, and

$$J(\lambda) = \frac{1}{2} \sum_{\substack{(K,L)\in\mathcal{I} \\ K>L}} \left\{ |U_K\lambda_K - U_L\lambda_L|_{0,\Gamma_{KL}}^2 + 2\int_{\Gamma_{KL}} (U_K\lambda_K - U_L\lambda_L)(\beta_K - \beta_L) + |\beta_K - \beta_L|_{0,\Gamma_{KL}}^2 \right\}$$

that is,

$$(3.5) \quad J(\lambda) = \frac{1}{2}\sigma(\lambda, \lambda) + \sum_{\substack{(K,L)\in\mathcal{I} \\ K>L}} \int_{\Gamma_{KL}} (U_K\lambda_K - U_L\lambda_L)(\beta_K - \beta_L) + \frac{1}{2} \sum_{(K,L)\in\mathcal{I}} |\beta_K - \beta_L|_{0,\Gamma_{KL}}^2,$$

where, for $\lambda, \mu \in \Lambda_-$ we have set

$$\sigma(\lambda, \mu) = \sum_{\substack{(K,L)\in\mathcal{I} \\ K>L}} \int_{\Gamma_{KL}} (U_K\lambda_K - U_L\lambda_L)(U_K\mu_K - U_L\lambda_L)$$

Lemma 3.1: The bilinear mapping σ is an inner product in $\tilde{\Lambda}_-$.

Proof: It is obvious that σ is bilinear and symmetric. That $\sigma(\lambda, \lambda) = 0 \Rightarrow \lambda = 0$ follows from (ii) $\Rightarrow$ (iii) in Theorem 3.1 for the special case when $f = 0$ (hence $u = 0$) and $\bar{u} = 0$ in (3.2). ☐

The norm in $\tilde{\Lambda}_-$ induced by the inner product $\sigma(\cdot, \cdot)$ will henceforth be referred to as the "σ-norm". It is easily seen that $\tilde{\Lambda}_-$ cannot be complete for the σ-norm, and hence we shall call Λ_-^σ the completion of $\tilde{\Lambda}_-$ for the σ-norm. From the trace theorems, there is a constant $C > 0$ such that

$$\sigma(\lambda, \lambda)^{1/2} \leq C \, \|\lambda\|_{*,\mathcal{P}}, \quad \forall \lambda \in \tilde{\Lambda}_-,$$

i.e. the σ-norm is weaker than the $\|\cdot\|_{*,\mathcal{P}}$ norm in $\tilde{\Lambda}_-$.

Theorem 3.2: The functional J is continuous and coercive in $\tilde{\Lambda}_-$ equipped with the σ-norm. As a result, it can be extended as a continuous and coercive quadratic functional in Λ_-^σ with the same minimum value (that is, 0). In particular, $\lambda = (\partial u_K / \partial \nu_A)$ remains the unique minimizer of J in Λ_-^σ.

Proof: Continuity and coercivity of J in $\tilde{\Lambda}_-$ for the σ norm is trivial from (3.5). That it can be extended to Λ_-^σ with the same minimum value is also trivial (using denseness of $\tilde{\Lambda}_-$ in its completion Λ_-^σ). It thus remains nonnegative and has a unique minimizer in Λ_-^σ, and this minimizer must be $(\partial u_K / \partial \nu_A)$ since $J((\partial u_K / \partial \nu_A)) = 0$. ☐

We now face the key problem in following the above approach: In practice, the minimizer $(\partial u_K / \partial \nu_A)$ of J will be obtained via minimizing sequences $\lambda^j \in \Lambda_-^\sigma$. By denseness of $\tilde{\Lambda}_-$ in Λ_-^σ, such sequences can be chosen in $\tilde{\Lambda}_-$ (and even in Λ_- by denseness of Λ_- in $\tilde{\Lambda}_-$), and for each index j we obtain an element $\phi(\lambda^j) \in H_0^1(\mathcal{P})$. It is easily seen that $\phi(\lambda^j)$ need not approach $u - \bar{u}$ in $H_0^1(\mathcal{P})$ as j tends to ∞. This is related to the fact that the σ-norm is too weak to ensure continuity of the bilinear form

$$(\lambda, v) \in \Lambda_- \times H_0^1(\mathcal{P}) \longmapsto \sum_{K=1}^{N} (\lambda_K, v_K)$$

when $\bar{\Lambda}_-$ is equipped with the σ-norm. Thus, in practice, minimizing the functional J using minimizing sequences (the only possible option snce $(\partial u_K/\partial \nu_A)$ is of course unknown) will not necessarily provide the desired result, i.e. will not yield an approximation on $u - \hat{u}$ in $H_0^1(\mathcal{P})$. Fortunately, this difficulty can be overcome because of the following result.

Theorem 3.3: Suppose that $u \in H^s(\Omega)$ for some $s > 3/2$. Then:

(i) There are minimizing sequences (λ^j) of J such that $\lambda^j \in \Lambda_-$ and (λ^j) is bounded in Λ_- (that is, bounded in $L^2(\Gamma_\mathcal{I})$).

(ii) If (λ^j) is a minimizing sequence of J such that $\lambda^j \in \Lambda_-$ and (λ^j) is bounded in Λ_-, we have

$$\lim_{j \to \infty} \|\phi(\lambda^j) - (u - \hat{u})\|_{1,\mathcal{P}} = 0.$$

Proof: If $u \in H^s(\Omega)$ for $s > 3/2$, then $\partial u_K/\partial \nu_A \in H^{s-\frac{3}{2}}(\Gamma_K) \subset L^2(\Gamma_K)$, $1 \leq K \leq N$. Thus, in particular, $(\partial u_K/\partial \nu_A) \in L^2(\Gamma_\mathcal{I})$. Also, if $u \in C^\infty(\bar{\Omega})$, it is obvious that $(\partial u_K/\partial \nu_A) \in \Lambda_-$, and this relation remains valid for $u \in H^s(\Omega)(s > 3/2)$ by denseness of $C^\infty(\Omega)$ into $H^s(\Omega)$, continuity of the mapping $v \in H^s(\Omega) \longmapsto (\partial v_K/\partial \nu_A) \in L^2(\Gamma_\mathcal{I})$ and closedness of Λ_- in $L^2(\Gamma_\mathcal{I})$. Existence of minimizing sequences (λ^j) with $\lambda^j \in \Lambda_-$ and (λ^j) bounded in Λ_- follows at once from this result (choose e.g. $\lambda^j = (\partial u_K/\partial \nu_A)$, $\forall j$).

Next, let (λ^j) be any minimizing sequence of J such that $\lambda^j \in \Lambda_-$ and (λ^j) is bounded in Λ_-. By compactness of the embedding $L^2(\Gamma_K \setminus \Gamma) \hookrightarrow V_K'$, there is $\lambda \in (H_0^1(\mathcal{P}))' = \prod_{K=1}^{N} V_K'$ and a subsequence (λ^{j_k}) such that $\lim_{k \to \infty} \|\lambda^{j_k} - \lambda\|_{*,\mathcal{P}} = 0$. Thus, $\lambda \in \bar{\Lambda}_-$ by definition of the space $\bar{\Lambda}_-$. Since the functional J is continuous for the σ-norm (see Theorem 3.2), hence for the stronger $\|\cdot\|_{*,\mathcal{P}}$ norm, we have $J(\lambda) = \lim_{k \to \infty} J(\lambda^{j_k})$, so that λ minimizes J. It follows that $\lambda = (\partial u_K/\partial \nu_A)$ (see Theorem 3.2).

The above reveals that $(\partial u_K/\partial \nu_A)$ is the unique cluster point of the sequence (λ^j) in $\bar{\Lambda}_-$, so that the whole sequence (λ^j) converges to $(\partial u_K/\partial \nu_A)$ in $\bar{\Lambda}_-$. This amounts to saying that λ_K^j tends to $\partial u_K/\partial \nu_A$ in V_K' for every $1 \leq K \leq N$. Now, from (3.3) we have

$$a_K(\phi_K(\lambda_K^j) - (u - \hat{u}), v_K) \leq \|\lambda_K^j - \frac{\partial u_K}{\partial \nu_A}\|_{*,\Omega_K} \|v_K\|_{1,\Omega_K}, \quad \forall v_K \in V_K.$$

Taking the supremum over $v_K \in V_K$, $\|v_K\|_{1,\Omega_K} \leq 1$, and using the fact that $a_K(\cdot,\cdot)$ satisfies the conditions (1.1a/b), we get

$$\|\phi_K(\lambda_K^j) - (u - \hat{u})\|_{1,\Omega_K} \leq C_K \|\lambda_K^j - \frac{\partial u_K}{\partial \nu_A}\|_{*,\Omega_K},$$

where $C_K > 0$ is a constant independent of j. Thus,

$$\left(\sum_{K=1}^{N} \|\phi_K(\lambda_K^j) - (u - \hat{u})\|_{1,\Omega_K}^2\right)^{1/2} \leq C\left(\sum_{K=1}^{N} \|\lambda_K^j - \frac{\partial u_K}{\partial \nu_A}\|_{*,\Omega_K}^2\right)^{1/2},$$

with $C = \max_{1 \leq K \leq N} C_K$, and the right-hand side is just $C\|\lambda^j - \left(\frac{\partial u_K}{\partial \nu_A}\right)\|_{*,\mathcal{P}}$ and hence tends to 0 as j tends to ∞ as noted earlier in the proof. □

By a method of proof similar to that of Thoerem 3.3, we obtain the following result without any extra assumption of regularity for the solution u:

Theorem 3.4: (i) There are minimizing sequences (λ^j) of J with $\lambda^j \in \bar{\Lambda}_-$ and (λ^j) bounded in $\bar{\Lambda}_-$.

(ii) If (λ^j) is a minimizing sequence of J such that $\lambda^j \in \bar{\Omega}$ and (λ^j) is bounded in $\bar{\Omega}$, we have

$$\lim_{j \to \infty} \|\phi(\lambda^j) - (u - \hat{u})\|_{r,\mathcal{P}} = 0,$$

for every $0 \leq r < 1$, where $\|\cdot\|_{r,\mathcal{P}}$ is the product norm of the space $H^r(\mathcal{P}) = \prod_{K=1}^{N} H^r(\Omega_K)$.

Proof: (i) is trivial by considering the constant sequence $(\partial u_K / \partial \nu_A)$. (ii) Let (λ^j) be a minimizing sequence in $\bar{\Lambda}_-$ which is bounded in $\bar{\Lambda}_-$, and let λ be a cluster point of (λ^j) for the weak topology of Λ_-, (Λ_- is a closed subspace of the Hilbert space $\prod_{K=1}^{N} V_K'$). If (λ^{j_k}) is a subsequence tending to λ, we still have $J(\lambda) = \lim_{k \to \infty} J(\lambda^{j_k}) = 0$ because J is convex and continuous for the $\|\cdot\|_{*,\mathcal{P}}$ norm, hence weakly lower semicontinuous. Thus, once again, $\lambda = (\partial u_K / \partial \nu_A)$ and the whole sequence (λ^j) tends weakly to $(\partial u_K / \partial \nu_A)$. At this point, note that together with this result, (3.3) implies that $\phi_K(\lambda_K^j)$ tends weakly to $(u - \hat{u})_{|\Omega_K}$ in V_K. If $L_K \in GL(V_K)$ is defined by

$$a_K(v_K, w_K) = (L_K v_K, w_K), \quad \forall v_K, w_K \in V_K,$$

where $(\cdot,\cdot)$ denotes the inner product of $H^1(\Omega_K)$, we have $\phi_K(\lambda_K^j) - (u - \hat{u})_{|\Omega_K} = L_K g_K^j$ where $g_K^j \in V_K$ is defined by

$$(g_K^j, v_K) = \langle \lambda_K^j - \frac{\partial u_K}{\partial \nu_A}, v_K \rangle, \quad \forall v_K \in V_K$$

That $g_K^j \to 0$ in V_K follows from $\lambda_n^j - \partial u_K/\partial \nu_A \to 0$ in V_K' and hence $\phi_K(\lambda^j) \to (u - \hat{u})_{|\Omega_K}$ by continuity of L_K.

The theorem now follows from compactness of the embedding $H^r(\Omega_K) \hookrightarrow H^1(\Omega_K)$ for $0 \leq r < 1$, $1 \leq K \leq N$. $\square$

The practical aspects of Theorems 3.3 and 3.4 is captured by the following corollary.

Corollary 3.2: Let $B \subset \tilde{\Lambda}_-$ be any closed ball such that $(\partial u_K/\partial \nu_A) \in B$. Then, B is closed in $\tilde{\Lambda}_-^\sigma$, $\inf_{\lambda \in B} J(\lambda) = \inf_{\lambda \in \tilde{\Lambda}_-^\sigma} J(\lambda)$ and if $\lambda^j \in B$ is a minimizing sequence of $J_{|_B}$, we have

$$\lim_{j \to \infty} \|\phi(\lambda^j) - (u - \hat{u})\|_{r,\mathcal{P}} = 0,$$

for every $0 \leq r < 1$.

Furthermore, if $u \in H^s(\Omega)$ for some $s > 3/2$ and $B \subset \Lambda_-$ is any closed ball such that $(\partial u_K/\partial \nu_A) \in B$, then B is closed in $\tilde{\Lambda}_-^\sigma$, $\inf_{\lambda \in B} J(\lambda) = \inf_{\lambda \in \tilde{\Lambda}_-^\sigma} J(\lambda)$ and if $\lambda^j \in B$ is a minimizing sequence of $J_{|_B}$, we have

$$\lim_{j \to \infty} \|\phi(\lambda^j) - (u - \hat{u})\|_{1,\mathcal{P}} = 0.$$

Proof: That B is closed in $\tilde{\Lambda}_-^\sigma$ follows from weak compactness of B in its ambient space: If $\lambda^j \in B$ and $\lambda^j \to \lambda \in \tilde{\Lambda}_-^\sigma$ in $\tilde{\Lambda}_-^\sigma$, let $\mu \in B$ be such that $\lambda^{j_k} \rightharpoonup \mu$ in $\tilde{\Lambda}_-$ or Λ_-. In both cases, this implies $\lambda^{j_k} \rightharpoonup \mu$ in $\tilde{\Lambda}_-^\sigma$ since the embeddings $\Lambda_- \hookrightarrow \tilde{\Lambda}_- \hookrightarrow \tilde{\Lambda}_-^\sigma$ are continuous, whence $\mu = \lambda$ and $\lambda \in B$. That the infimum of J is the same in B and in the whole space $\tilde{\Lambda}_-^\sigma$ is trivial from $(\partial u_K/\partial \nu) \in B$ being the unique minimizer of J in $\tilde{\Lambda}_-^\sigma$. Convergence of $\phi(\lambda^j)$ to $u - \hat{u}$ follows from Theorems 3.3 and 3.4. $\square$

In practical applications, the hypothesis that some ball B containing $(\partial u_K/\partial \nu_A)$ is known is not a severe limitation since some useful information can be derived from the approximate solution $\hat{u}$. See Section 4.

4. Practical Aspects.

In practice, the appoximate solution $\hat{u}$ will be a finite element approximation $u^h \in V^h$ where V^h is a finite element subspace of $H_0^1(\Omega)$ and the partition $\mathcal{P}$ corresponds to the partition of Ω into elements. It is certainly not restrictive to assume that u^h is a "reasonable" approximation of u, as opposed to a completely random element of V^h, and hence that despite the fact that $\|u - u^h\|_{1,\Omega}$ may not be small, or small enough, still the derivative $(\partial u_K^h/\partial \nu_A)$ gives some idea of where in $\bar{\Lambda}_-$ (or Λ_- if $u \in H^s(\Omega)$, $s > 3/2)(\partial u_K/\partial \nu_A)$ is. One problem is that, in general, $(\partial u_K^h/\partial \nu_A) \notin \Lambda_-$, but this is easily remedied by replacing $(\partial u_K^h/\partial \nu_A)$ by say, $(\frac{\partial u_K}{\partial \nu_A})_{1/2} \equiv (\frac{1}{2}(\partial u_K^h/\partial \nu_A - \partial_L^h/\partial \nu_A))_{(K,L)\in \mathcal{I}}$. *Other weighted averages, as suggested in [ ] can also be used.*

The ball B of Corollary 3.2 can be chosen as any ball with center $\langle \partial u/\partial \nu_A \rangle_{1/2}$ and arbitrary radius $R > 0$. Because of denseness of Λ_- in $\bar{\Lambda}_-$ and $\bar{\Lambda}_-^e$, finite-dimensional approximations of the space Λ_-^e can be obtained by choosing a scale Λ_-^j of finite-dimensional subspaces of Λ_-. One possible choice is given by collections $\lambda = (\lambda_K)$ such that λ_K is a polynomial of degree $\leq j$ on each face of Ω_K not lying on the boundary of Ω. It should be kept in mind that membership to Λ_- requires the condition $\lambda_{K|_{\Gamma_{KL}}} = -\lambda_{L|_{\Gamma_{KL}}}$, and hence λ_K and λ_L cannot be chosen independently. Because no continuity condition is required of λ_K at the intersection of two faces, bases for such spaces are easily found. For instance, if $j = 0$, a basis is given by the collections (λ_{KL}) where $\lambda_{KL} = 0$ if $(K, L) \neq (K_0, L_0)$ where $(K_0, L_0) \in \mathcal{I}$ satisfies $K_0 > L_0$, $\lambda_{K_0L_0} = 1$, $\lambda_{L_0K_0} = -1$, letting (K_0, L_0) run over all such pairs. It should be clear how this procedure can be extended to obtain bases for arbitrary polynomial degree j.

One may then define λ^j to be the minimizer of J in $B \cap \Lambda_-^j$, a closed convex subset of Λ_-^j. The sequence (λ^j) is a minimizing sequence for J in B, and hence the conclusion of Corollary 3.2 regarding convergence of $\phi(\lambda^j)$ to $u - u^h (= u - \hat{u}$ here) applies. Rather than minimizing J in $B \cap \Lambda_-^j$ – 0 a constrained problem – it seems advisable to minimize J in Λ^j (a linear problem) and check whether this minimizer is in B, and modify the choice of the minimizer only if this is not the case (i.e. the minimizer is unreasonably far from $\langle \partial u_K^h/\partial \nu_A \rangle_{1/2}$).

Naturally, replacing Λ_- by a finite dimensional approximation Λ_-^j is not enough, since

$\phi(\lambda)$ cannot be calculated exactly. However, since $\phi(\lambda) = (\phi_K(\lambda_K))$ is obtained by solving independent local problems, numerical approximations can be obtained at a low cost with high accuracy. In other words, the error between $\phi(\lambda)$ and its numerical approximation may be considered negligible when compared with $\|u-u^h\|_{1,\Omega}$, and hence the convergence result of Corollary 3.2 may be used safely with $\phi(\lambda)$ replaced by its numerical approximation.

Remark 4.1: When the spaces Λ^j_- are those previously described, i.e. their elements are collections of polynomials with degree $\leq j$ on the interfaces, it is desirable that the space chosen for the approximation of $\phi_K(\lambda_K)$ when $\lambda = (\lambda_K) \in \Lambda^j_-$, contains the polynomials of degree $\leq j+1$ in Ω_K. Indeed, λ_K is used to obtain an approximation of $\partial u_K/\partial\nu_A$ whereas $\phi_K(\lambda_K)$ represents an approximation of $(u-u^h)_{|\Omega_K}$, and hence polynomials with degree at least $j+1$ should be used to approximate $\phi_K(\lambda_K)$ for consistency (assuming $A = (a_{ij})$ essentially constant on each element). The above considerations also show that $j+1$ should be at least equal to the degree of the polynomials used for the calculation of u^h, which comes as no surprise. □

5. Nonlinear problems.

Let $a(\cdot,\cdot), a_K(\cdot,\cdot)$ be the same bilinear forms as in Section 3, and let $F : \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ be a mapping such that $F(\cdot,\cdot)$ is of class C^1 for almost all $x \in \Omega$ and $F(\cdot,y)$ is measurable for every $y \in \mathbf{R}$. We also assume that there are constants $\beta \geq 0, \gamma \geq 0$ such that

$$|D_yF(x,y)| \leq \beta + \gamma|y|^{p-2}$$

for almost all $x \in \Omega$ and every $y \in \mathbf{R}$, where $p > 2$ is a real number satisfying $p \leq \frac{2n}{n-2}$ if $n > 2$. This implies that the mapping

$$\tilde{F} : u \in L^p(\Omega) \Rightarrow F(u) \in L^{p^*}(\Omega)\ (p^* = p/(p-1))$$

defined by $\tilde{F}(u)(x) = F(x,u(x))$ is of class C^1 with derivative

$$D\tilde{F}(u)h = D_yF(x,u(x))h(x).$$

It then follows from Taylor's formula that

$$|\tilde{F}(u) - \tilde{F}(v)|_{0,p^*,\Omega} \leq C((\text{meas } \Omega)^{\frac{p-2}{p}} + \gamma(|u|_{0,p,\Omega}^{p-2} + |v|_{0,p,\Omega}^{p-2})|u - v|_{0,p,\Omega},$$

for every pair $(u,v) \in (L^p(\Omega))^2$, where $C > 0$ is a constant independent of u and v.

This shows that if $u^h \in V^h$ is a finite element approximation of $u \in H_0^1(\Omega)$ such that

$$\lim_{h \to 0} \|u - u^h\|_{1,\Omega} = 0, \quad \lim_{h \to 0} \frac{|u - u^h|_{0,p,\Omega}}{\|u - u^h\|_{1,\Omega}} = 0,$$

we have

$$(5.1) \qquad |\tilde{F}(u) - \tilde{F}(u^h)|_{0,p^*,\Omega} \leq \epsilon(h)\, \|u - u^h\|_{1,\Omega},$$

with $\lim_{h \to 0} \epsilon(h) = 0$.

Suppose now that u solves the problem

$$a(u,v) + \int_\Omega \tilde{F}(u)v = \int_\Omega fv \quad , \quad \forall v \in H_0^1(\Omega),$$

i.e. $u \in H_0^1(\Omega)$ and

$$-\nabla \cdot (A \nabla u) + b \cdot \nabla u + cu + \tilde{F}(u) = f \quad \text{in } \Omega.$$

In analogy with what was done in Section 3, let $\lambda \in \tilde{\Lambda}_-$ and solve, for $1 \leq K \leq N$,

$$\begin{cases} a_K(\phi_K(\lambda_K), v_K) = \int_{\Omega_K} f v_K - a_K(u^h, v_K) - \int_{\Omega_K} F(u^h) v_K + \langle \lambda_K, v_K \rangle \\ \forall v_K \in V \end{cases}$$

that is,

$$(5.2) \quad \begin{cases} a_K(\phi_K(\lambda_K), v_K) = a_K(u - u^h, v_K) + \int_{\Omega_K} (\tilde{F}(u) - \tilde{F}(u^h)) v_K + \langle \lambda_K - \frac{\partial u_K}{\partial \nu_A}, v_K \rangle \\ \forall v_K \in V \end{cases}$$

By the method of proof of Theorem 3.1, it is easily seen that the only case when $\phi(\lambda) = (\phi_K(\lambda_K)) \in H_0^1(\Omega)$ is when $\phi(\lambda)$ coincides with the solution $\psi^h \in H_0^1(\Omega)$ (not V^h) of

$$a(\psi^h, v) = a((u-u^h), v) + \int_\Omega (\tilde{F}(u) - \tilde{F}(u^h))v, \quad \forall v \in H_0^1(\Omega), \tag{5.3}$$

and that if so, $\lambda_K = \partial(\psi^h + u^h)_K/\partial\nu_A$, $1 \le K \le N$. In fact, in terms of ψ^h above, (5.2) reads

$$\begin{cases} a_K(\phi_K(\lambda_K), v_K) = a_K(\psi^h, v_K) + \langle \lambda_K - \dfrac{\partial(\psi^h + u^h)_K}{\partial\nu_A}, v_K\rangle \\ \forall v_K \in V_K. \end{cases}$$

This relation can be used to show that if (λ^j) is a minimizing sequence of the functional J of Section 3 (which now has unique minimizer $(\partial(\psi^h + u^h)_K/\partial\nu_A)$) and if (λ^j) is bounded in Λ_-, then

$$\lim_{j\to\infty} \|\phi(\lambda^j) - \psi^h\|_{1,\mathcal{P}} = 0. \tag{5.4}$$

(And if (λ^j) is only bounded in $\tilde{\Lambda}_-$,

$$\lim_{j\to\infty} \|\phi(\lambda^j) - \psi^h\|_{r,\mathcal{P}} = 0 \quad , \quad \forall 0 \le r < 1.)$$

To obtain an estimate for $\|\phi(\lambda^j) - (u-u^h)\|_{1,\mathcal{P}}$, write

$$\|\phi(\lambda^j) - (u-u^h)\|_{1,\mathcal{P}} \le \|\phi(\lambda^j) - \psi^h\|_{1,\mathcal{P}} + \|(u-u^h) - \psi^h\|_{1,\Omega}. \tag{5.5}$$

From (5.3),

$$a(\psi^h - (u-u^h), v) = \int_\Omega (\tilde{F}(u) - \tilde{F}(u^h))v.$$

Taking the supremum over $v \in H_0^1(\Omega)$, $\|v\|_{1,\Omega} \le 1$ and using the fact that $a(\cdot,\cdot)$ satisfies (1.1a/b), we get

$$a\|\psi^h - (u-u^h)\|_{1,\Omega} \le \sup_{\|v\|_{1,\Omega}\le 1} \int_\Omega (\tilde{F}(u) - \tilde{F}(u^h))v,$$

where $c > 0$ is a constant independent of u and h. Next, using $|\int_\Omega (\tilde{F}(u) - \tilde{F}(u^h))v| \leq |\tilde{F}(u) - \tilde{F}(u^h)|_{0,p^*,\Omega} |v|_{0,p,\Omega}$ and continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, we obtain

$$\|\psi^h - (u - u^h)\|_{1,\Omega} \leq C|\tilde{F}(u) - \tilde{F}(u^h)|_{0,p^*,\Omega},$$

where $C > 0$ is a constant independent of u and h. Finally, using (5.1) we see that

$$\|\psi^h - (u - u^h)\|_{1,\Omega} \leq C\epsilon(h)\|u - u^h\|_{1,\Omega}.$$

Substituting into (5.5), we get

$$\|\phi(\lambda^j) - (u - u^h)\|_{1,\mathcal{P}} \leq \|\psi(\lambda^j) - \psi^h\|_{1,\mathcal{P}} + C\epsilon(h)\|u - u^h\|_{1,\Omega}.$$

and, in particular, if (5.4) holds:

$$\lim_{j \to \infty} \|\phi(\lambda^j) - (u - u^h)\|_{1,\mathcal{P}} \leq C\epsilon(h)\|u - u^h\|_{1,\Omega}, \tag{5.6}$$

i.e. $\|\psi(\lambda^j) - (u - u^h)\|_{1,\mathcal{P}}$ becomes negligible with respect to $\|u - u^p\|_{1,\Omega}$ when $\phi(\lambda^j)$ is a good enough approximation of ψ^h because $\lim_{h \to 0} \epsilon(h) = 0$.

Remark 5.1: From (5.6), it also follows that $u^h + \phi(\lambda^j)$ is a better (enhanced) approximation of u than u^h. Interestingly, this enhanced solution is obtained by solving only *linear* problems (calculation of $\phi(\lambda^j)$). □

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